COMMENTS TO IRREVERSIBILITY IN THERMODYNAMICS

Michail Zak
Jet Propulsion Laboratory
California Institute of Technology
Pasadena, CA 91109

ABSTRACT

The problem of irreversibility in thermodynamics was revisited and analyzed on the microscopic, stochastic, and macroscopic levels of description, It was demonstrated that Newtonian dynamics can be represented in the Reynolds form when each dynamical variable is decomposed into the mean and fluctuation components. Additional equations coupling fluctuations and the mean values follow from the stabilization principle. The main idea of this principle is that the flue.tuati(ms must be selected from the condition that they suppress the original instability down to a neutral stability. Supplemented by the stabilization principle, the Hamiltonian, or Largranian formalisms can describe the transition from fully reversible to irreversible motions as a result of the decomposition of chaotic motions (which are very likely to occur in many-body problems) into regular (macroscopic) motions and fluctuations.

On the stochastic level of description, new phenomenological force with non-Lipschitz properties is introduced. This force as a resultant of a large number of collisions of a selected particle with other particles, has characteristics which are uniquely defined by the thermodynamical parameters **of** the process under consideration, and it represents a part of the mathematical formalism describing random-walk-like process without invoking any probabilistic. arguments.

Additional non- Lipscitz thermodynamical forces were incorporated into macroscopic models of transport phenomena in order to introduce a time scale. These forces are effective only within a small domain around equilibria. Without causing any changes in other domains, they are responsible for finite time of approaching equilibria, Such a property is very important for interpretation of irreversibility on the macroscopic scale.

INTRODUCTION:

Transport phenomena such as thermal conductivity and diffusion represent non equilibrium thermodynamical processes which are described by parabolic partial differential equations of the following type:

$$\frac{\partial u}{\partial t} = \mathcal{D}_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}, \ \mathcal{D}_{ij} = \text{const} \tag{1}$$

It is known that Eq. (1) subject to the initial condition

$$u \mid_{t=0} = u_0(x) \tag{2}$$

has a unique bounded solution for $t > 0$.

However, for $t < 0$ the same problem is ill-posted, and that expresses the fundamental property of irreversibility of thermal conductivity and diffusion. Actually this property directly follows from the second law of thermodynamics.

Although solutions to Eq. (1) are in sufficiently good agreements with experiments, there are still some logical difficultures in reconciliation of this macroscopical phenomenological model with the fully reversible Hamiltonian dynamics on the microscopic level, since, actually the irreversible processes described by Eq. (1), are completely composed of reversible events, and that is known as the irreversibility paradox. However, strictly speaking, the formal derivation of Eq. (1) from the microscopic Hamiltonian mechanics requires some additional arguments of a probabilistic nature. But can these arguments be represented in terms of classical mechanics"? Or, more precisely, can they be replaced by some equivalent mechanical forces on the microscopic level?

1. NON-LIPSCHITZ MECHANICS

Turning to governing equations of classical dynamics:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} - \frac{\partial R}{\partial \dot{q}_i}, \quad i = 1, 2, \ldots n \tag{3}$$

where L is the Lagrangian, $q_i, \dot{q}_i$ are the generalized coordinates and velocities, and R is the dissipation function, one should recall that the structure of $R(\dot{q}_i, \ldots \dot{q}_n)$ is not prescribed by Newtons laws: some additional assumptions are to be made in order to define it. The "natural" assumption (which has been never challenged) is that these functions can be expanded in Taylor series with respect to equilibrium states: $\dot{q}_i = 0$. Obviously this requires the existence of the derivative:

$$\left| \frac{\partial^2 R}{\partial \dot{q}_i \, \partial \dot{q}_i} \right| < \infty \text{ at } \dot{q}_i \to 0$$

The departure from that condition was proposed in [8-10], where the following dissipation function was introduced:

$$R = \frac{1}{k+1} \sum_i \alpha_i \left| \sum_j \frac{\partial r_i}{\partial q_j} \dot{q}_j \right|^{k+1} \tag{4}$$

in which

$$k = \frac{p}{p+2} < 1, \; p \gg 1 \tag{5}$$

while p is a large odd number

By selecting large p, one can make k close to 1 so that Eq. (4) is almost identical to classical one (when $k = 1$) everywhere excluding a small neighborhood of the. equilibrium point $\dot{q}_j = 0$, while at this point:

$$\left| \frac{\partial^2 R}{\partial \dot{q}_i \partial \dot{q}_j} \right| \to \infty \text{ at } \dot{q}_j \to 0 \tag{6}$$

Hence, the Lipschitz condition is violated, the friction force $F_i = -\frac{\partial R}{\partial \dot{q}_i}$ grows sharply at the equilibrium point, and then it gradually approaches its "classical" value. This effect can be interpreted as a mathematical representation of a jump from static to kinetic friction, when the dissipation force does not vanish with the velocity.

It appears that this "small" difference between the friction forces at $k = 1$ and $k < 1$ leads to fundamental changes in Newtonian dynamics. In order to demonstrate it, we will consider the relationship between the total energy E and the dissipation function R:

$$\frac{dE}{dt} = -\sum_i \dot{q}_i \frac{\partial R}{\partial \dot{q}_i} = -(k+1)R \tag{7}$$

Within a small neighborhood of an equilibrium state (where the potential energy can be set zero) the energy E and the dissipation function R have the order, respectively:

$$E \sim \dot{q}_i^2, \; R \sim \dot{q}_i^{k+1} \text{ at } E \to 0 \tag{8}$$

Hence, the asympotatical form of (7) can be presented as:

$$\frac{dE}{dt} = AE^{\frac{k+1}{2}} \text{ at } E \to 0, \; A = \text{const} \tag{9}$$

If $A > 0$ and $k < 1$, the equilibrium state $E = O$ is an attractor where the. Lipschitz condition ($|d\dot{E}/dE| \to \infty$ at $E \to O$) is violated. Such a terminal [8] attractor is approached by the solution originated at $E = \Delta E_o > 0$, in finite time:

$$t_o = \int_{\Delta E_o}^{o} \frac{dE}{AE^{\frac{k+1}{2}}} = \frac{2 \Delta E_o^{\frac{1-k}{2}}}{(1 - k) \mid A \mid} < \infty$$

Obviously, this integral diverges in classical cast? $k \geq 1.$, where $t_o \to \infty$. The motion described by (9) has a singular solution $E \equiv O$ *and* a regular solution:

$$E = \left[\Delta E_o^{\frac{1-k}{2}} + \frac{1}{2} A(1 - k)t\right]^{\frac{2}{1-k}}$$

In a finite time the motion can reach the equilibrium and switch to the singular solution $E \equiv 0$, *and* this switch is irreversible.

As well-known from dynamics of non-conservative systems, dissipative forces can destabilize the motion when they feed the external energy into the system (the transmission of energy from laminar to turbulent flow in fluid dynamics, or from rotations to oscillations in dynamics of flexible systems). In terms of (9) it would mean that $A > 0$, and the equilibrium state $E = O$ becomes a terminal repeller [8].

If the initial condition is infinitely close to this repeller, the transient solution will escape it during a finite time period:

$$to \quad \int_{\varepsilon \to 0}^{\Delta E_o} \frac{dE}{A\ E\ \%} = \frac{2 \Delta E_o^{\frac{1-k}{2}}}{(1 - k)A} =< \infty$$

while for a regular repeller, the time would be infinite.

Expressing Eq. (9) in terms of velocity at $i = 1$, $\dot{q}_1 = v$,

$$\dot{v} = Bv^k, B = \text{const} > (), \tag{1o}$$

one arrives at the following solution:

$$v = \pm\left\{ [B(1 - k)t]^{p+2}\right\}^{1/2} \tag{11}$$

As in the case of a terminal attractor, here the motion is also irreversible: the time-backward motion obtained by formal time inversion $t \to -t$ in Eq. (11) is imaginary, since p is an odd number (see Eq. (5)).

But in addition to that, terminal repellers possess even more surprising characteristics: the solution (11) becomes totally unpredictable. Indeed, two different motions described by the solution (11) are possible for "almost the same" ($v_o = +\varepsilon \to 0$, or $v_o = -\varepsilon \to 0$ at $t = \to 0$) initial conditions.

Thus, a terminal repeller represents a vanishingly short, but infinitely powerful "pulse of unpredictab ility" which is pumped into the syst em via tern iinal dissipative? forces. Obviously failure of the uniqueness of the solution here results from the violation of the Lipschitz condition at $v = 0$.

Hence, the non-Lipschitz forces $\partial R/\partial q_i$ in Eq. (3) following from Eqs. (4) and (5) change the most fundamental property of the Newtonian mechanics: its determinism. At the same time, these forces affect only the dissipation function which is not prescribed by t he Newt on 'S laws any way.

Let us turn to stochastic processes which connect the microscopical mechanics and thermodynamics. These processes are based upon some probabilistic arguments which can not be formall y derived from Newtonian lntdm.nits. But may be they can be derived from non- Lipschitzian version of Newtonian mechanics? In the ı ıext item based upon non-Lipschitz forces we will introduce a pure mechanical model of random walk-the simplest stochastic process - whose macroscopical interpret at ion leads t o Eq. (1).

2. MECHANICAL MODEL OF RANDOM WALK

A random walk is a stochastic process where changes occur only at fixed times; it represents the position at time t_m of a particle taking a random "step" x_m independently of its previous ones.

in order to implement this process based only upon tilt? Newton's laws, consider a rectilinear motion of a particle of unit mass driven by a non-Lipschitz force:

$$\dot{v} = \nu v^{1/3} \sin \omega t, \quad \nu = \text{const}, [\nu] = \frac{\mathrm{m}^{1-k}}{\sec^{2-k}} \tag{12}$$

$$\dot{x} = v \tag{13}$$

where v and x are the particle velocity and position, respectively.

Subject to zero initial condition:

$$v = 0 \quad \text{at} \quad t = 0, \tag{14}$$

Eq. (10) has a singular solution:

$$v \equiv 0 \tag{15}$$

and a regular solution:

$$v = \pm\left(\frac{4\nu}{3\omega}\sin^2\frac{\omega}{2}t\right)^{3/2} \tag{16}$$

These two solutions co-exist at $t = 0$, and that is possible because at this point the Lipschiz condition fails:

$$\left|\frac{\partial \dot{v}}{\partial v}\right|_{t\to 0} = k\nu v^{k-1}\sin\omega t|_{t\to 0} \quad \to \quad \infty \tag{17}$$

Since

$$\frac{\partial \dot{v}}{\partial v} > 0 \ \text{ at } \ |v| \neq 0, \quad t > 0, \tag{18}$$

the singular solution (15) is unstable, and the particle departs from rest following the solution (16). This solution has two (positive and negative) branches (since the power in (16) includes the square root), and each branch can be chosen with the same probability 1/2. It should be noticed that as a result of (1 7), the motion of the particle can be initiated by infinitesimal disturbances (that never can occur when the Lipschitz condition is in place: an infinitesimal initial disturbance cannot become finite in finite time).

Strictly speaking, the solution (16) is valid only in the tire? interval

$$0 \leq t \leq \frac{2\pi}{\omega}, \tag{19}$$

and at t = $\frac{2\pi}{\omega}$ it coincides with the singular solution (15).

For $t > 2\pi/\omega$, Eq. (15) becomes unstable, and the motion repeats itself to the accuracy of the sign in Eq. (16).

Hence, the particle velocity v performs oscillations with respect to its zero value in such a way that the positive and negative branches of the solution (16) alternate randomly after each period equal to $2\pi/\omega$.

Turning to Eq. (13), one obtains the distance between two adjacent equilibrium positions of the particle:

$$x_i - x_{i-1} = \pm \int_0^{2\pi/\omega} \left(\frac{4\nu}{3\omega} \sin \frac{\omega}{2} t \right)^{3/2} dt = 64(3\omega)^{-5/2} \nu^{3/2} = \pm h \tag{20}$$

Thus, the equilibrium positions of the particle are:

$$x_0 = 0, x_1 = \pm h, x_2 = \pm h \pm h . \text{etc.} \tag{22}$$

while the signs are randomly alternated with the equal probability 1/2.

Obviously, the particle preforms an unrestricted symmetric random walk: after each time period

$$T = \frac{2\pi}{w} \tag{23}$$

it changes its value on $\pm h$ (see Eq. (22)).

The probability density $u(x, t)$ is governed by the following difference equation:

$$u(x, t + \tau) = \frac{1}{2} u(x - h, t) + \frac{1}{2} u(x + h, t) \tag{24}$$

while

$$\int_{-\infty}^{\infty} u(x, t) dx = 1 \tag{25}$$

3. PHENOMENOLOGICAL FORCE

Thus, as demonstrated above, a non-Lipschitz force

$$F = m\nu v^{1/3} \sin \omega t = \pm \gamma \sqrt[3]{\frac{4\nu}{3\omega}} \sin \frac{\omega}{2} t \sin \omega t \tag{26}$$

applied to a particle of the mass m, leads to a classical random walk.

It should be stressed that the governing equations (12), (13) are fully deterministic: they are bad upon the Newton's laws. The stochasticity here is generated by the alternating stability and instability effects due to failure of the Lipschitz conditions at equilibria.

Let us analyze the properties of the force ('28).

First of all, the time average of this force is zero:

$$\tilde{F} = 0 \tag{27}$$

since, as follows from Eq. (26), the signs + and - have equal probability.

For the same reason, the ensemble average of F is also zero:

$$< F >= 0 \tag{28}$$

The work clone. by the force (26) during one step is zero:

$$A = \int_0^{2\pi/\omega} F v dt = \pm\nu\left(\frac{4\nu}{3\omega}\right)^2 \int_0^{2\pi/\omega} \sin^4 \frac{\omega}{2} t \sin \omega t dt = 0. \tag{29}$$

Sine.c the time average of the particle's kinetic energy can be expressed via the temperature, one obtains:

$$\tilde{v}^2 = \left(\frac{4\nu}{3\omega}\right)^3 \int_0^{2\pi/\omega} \sin^6 \frac{\omega}{2} t dt = \frac{5\pi}{8\omega}\left(\frac{4\nu}{3\omega}\right)^3 = \frac{KT}{m} \tag{30}$$

Then the only unspecified parameter ν in Eq. (26) is expressed via the. temperature:

$$\nu = \frac{3\omega}{4}\sqrt[3]{\frac{8\omega KT}{5\pi m}} \tag{31}$$

Here T is the absolute temperature, and K is the 1 3oltzmann's constant.

The parameter w $^{-1}$ has the order of the time period between collisions of the particle:

$$\omega \sim \frac{1}{\tau} \sim 10^{14} \frac{1}{\text{sec}} \tag{32}$$

On the macro-scale this is a very large number, and one can consider a centinuous approximation assuming that

$$\omega \to \infty \tag{33}$$

Then, as follows from Eqs. (20), (23), and (31) :

$$\tau \to \mathbf{0},\ \mathbf{11} \to \mathbf{o}, \text{ and } \frac{h^2}{\tau} \to \mathbf{0.19}\ \frac{KT}{m} = 2\mathcal{D} \tag{34}$$

and therefore, Eq. (24) can be replaced by the Fokker-Planck equation, i.e., by an one-dimensional version of Eq. (1). It is interesting to emphasize that the diffusion coefficient $\mathcal{D}$ is defined by the amplitude ν of the non-Lipschitz force (26).

Now the following question can be asked: does the force (26) exist in a sense that it can be detected by direct measurements on the mitxmc.epic level'? Probably, not. Indeed, on that level, this force is a resultant of a large number of collisions with other particles. However, on the stochastic level as an intermediate between the micro-and-macro-levels, the phenomenological force (26) represents a part of the mathematical formalism, and it can be accepted.

As follows from Eq. (26), on a micro-scale of time

$$t \sim \tau \tag{35}$$

the system (12), (13) is not conservative, and the motion is irreversible. Moreover, each time the particle arrives at equilibrium point, it totally "forgets)' its past.

On the contrary, on a macro-scale of time when

$$t >> \tau, \tag{36}$$

the system (12),(13) can be treated as conservative based upon Eqs. (28), and (29), and therefore, it is fully reversible. This means that the particle whose motion is described by Eq. (12) and (13), can return to its original position passing through all of its previous Steps backward; however, the probability of such an event will be vanishingly small (but not zero!), or, in other words, the period of time t_0 during which this event can occur is very large (but finite!):

$$\tau \ll t_0 < \infty \tag{37}$$

4. NON-LIPSCHITZ MACROSCOPIC EFFECTS:

Turning back to the macroscopic equation (1), one can notice its inconsistency with the results discussed in the last section, and in particular, with the condition (37). Indeed, Eq. (1) does not have any time scale which would allow to implement the condition (37): the time of approaching a thermodynamical equilibrium is unbounded, and therefore, Eq. (1) includes any reversible solutions even if $t_0 \to \infty$. The only logical way out of this situation is to introduce a time-scale to into Eq. (1) so that the time of approaching an equilibriun would be finite. Then one can argue that this time is not large enough to include reversible solutions. In order to do that, let us turn to Eq. (1), and, for the sake of cone.rdeness, treat it as an equation for thermal conductivity. Then, the relationship between the heat flow q and the temperature u can be sought in the following form:

$$q = q(\nabla u) \tag{38}$$

It should be (emphasized that the? function (3S) is not prescribed by any macroscopic laws, and therefore, it must be found from experiments. The basic mathematical assumption about Eq. (38) is its expendability in Taylor series. Then, for small gradients:

$$q = -\chi \nabla u + \cdots \text{etc.} \tag{39}$$

where χ is the thermal conductivity, and this leads to Eq. (1). But even if higher order gradients of u are taken into account, the time of approaching equilibria would still remain unbounded.

However, there is another possibility in representing Eel, (3S) if one relax the Lipschitz condition at $\nabla u = O$. Indeed, instead of (39) one can write:

$$q = -\chi \left(\frac{\nabla u}{\varepsilon_0}\right)^{k-1} \nabla u + \cdots \text{etc.} \tag{40}$$

where k has the form (5), and ε_0 has the dimensionality of ∇u, i.e.

$$[\varepsilon_0] = [\nabla u] \tag{41}$$

Eq. (40) is different from Eq. (39) only within an infinitesimally small neighborhood of the equilibria states where

$$\nabla u + \mathrm{O}. \tag{42}$$

otherwise

$$\left(\frac{\nabla u}{\varepsilon_0}\right)^{k-1} \simeq 1 \tag{43}$$

One can verify that the Lipsch tz condition for the function (40) at $\nabla u + \mathrm{O}$ is violated:

$$\left| \frac{\partial q}{\partial \nabla u} \right| \to \infty \quad \text{at} \quad \nabla u \to 0 \tag{44}$$

Mathematical consequences of this property will be discussed Mow.

Turning to Eq. (40), one can write the following equation instead of (1) :

$$\frac{\partial u}{\partial t} = \mathcal{D}\frac{\partial}{\partial x}\left[\left(\frac{\partial u}{\partial x}\right)^k\right], \quad \mathcal{D} = \frac{\chi \varepsilon_0^{1-k}}{\rho c} = \text{('(rest} > 0 \tag{45}$$

where χ, c, and p are the coefficient of thermal conductivity, specific heat, and density, respectively. Eq. (45) reduces to the classical diffusion equation:

$$\frac{\partial u}{\partial t} = \mathcal{D}\frac{\partial^2 u}{\partial x^2} \tag{46}$$

if $k = 1$.

Let us compare the solutions to Eqs. (45) and (46) subject to the same initial and boundary condit ions. Introducing the function:

$$\theta = \int \left(\frac{\partial u}{\partial x}\right)^{k-1} dx \tag{47}$$

one obtains:

$$\frac{d\theta}{dt} = (k+1)\int \left(\frac{\partial u}{\partial x}\right)^{k} \frac{\partial^2 u}{\partial x \partial t} dx = (k+1)\mathcal{D}\int \left(\frac{\partial u}{\partial x}\right)^{k} \frac{\partial^2}{\partial x^2}\left[\left(\frac{\partial u}{\partial x}\right)\right]^{k} dx$$

assuming separation of the variables:

$$u(x,t) = u_1(t)u_2(x) \tag{48}$$

one arrives at the following ordinary differential equation:

$$\dot{u}_1 = -Au_1^k \tag{49}$$

where

$$A = \mathcal{D}\int \left(\frac{\partial u_2}{\partial x}\right)^{k} \frac{\partial^2}{\partial x^2}\left[\left(\frac{\partial u}{\partial x}\right)^{k}\right] dx = \text{const} \tag{50}$$

For $k = 1$ (see Eq. (46)):

$$u_1 = \overset{o}{u}_1 \ \bar{e}^{At}, \quad u_1^2 \to 0 \text{ at } t \to \infty \tag{51}$$

For $k < 1$ (See Eq. 45):

$$u_1 = [(81)\ 1\text{-}'\ -A(I - k)t]^{1/1-k} \tag{52}$$

Here

$$\overset{o}{u}_1 = u_1 \quad \text{at} \quad t = 0 \tag{53}$$

5. MICROSCOPIC VIEW

In the previous sections, the problem of irreversibility in thermodynamics was discussed on the stochast ic and macroscopic levels of description. This and all the next sections will be devoted to the same problem, but from the viewpoint of the microscopic level of description. On that level, the microscopic state of a system may be specified in terms of positions and moments of a constituent set of particles: the atoms and moleculas. Within the Born-Oppenheimer approximation, it is possible to express the Hamiltonian (or the Lagrangian) of a system as a functions of nuclear variables, the (rapid) motions of elect rons having been averaged out. Making the additional approximation that a classical description is adequat e, one can write the Lagrange equat ions whi ch govern the microscopic motions of the system:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, 2, \cdots n, \quad L = W + \Pi \tag{59}$$

Here q_i and $\dot{q}_i$ are the generalized coordinates and velocities characterizing the system, W is the kinetic energy including translational compo nents (as well as rot ational components if polyatomic molecules are considered), n is the potential energy representing the effects of an external field (including, for example, the container walls), tht? particle interactions and elastic collisions.

All the solutions to Eqs. (59) are fully deterministic and reversible if the initial conditions are known exactly. But since the last requirement is physically unrealistic, small errors in initial conditions will grow exponentially in case of instability of Eels. (59). (Such an imitability may have the same origin as the instability in the famous three-body problem), As a result of that, the solution to Eq. (59) at tains stochastic features, i.e., becomes chaotic, and therefore, it looses its determinism and reversibilit y. The connection bet ween the chaotic inst ability and the. problem of irreversibili t y in thermodynamics was stressed by I. Progogine [5]: "The structure of the equations of motion with "randomness" on the microscopic level then emerges as irreversibility on the macroscopic level". Based upon the same ideas as those introduced by Prigogine, we will propose a different mathematical framework for their implement ati on. This framework exploits the stabilization principle introduced and discussed in [11]. As will be shown below, this principle imposes some additional constraints upon the motion, and that makes the solut ions to Eqs. (59) irreversible.

6. ORBITAL INSTABILITY IN HAMILTONIAN MECHANICS

Most of the dynamical processes are so complex that their universal theory which would capture all the details during all the time periods is unthinkable. That is why the art of mathematical modeling is to extract only tht? fundamental aspects of the process and to neglect its insignificant features, without losing the core of information. But "insignificant features" is not a simple concept. In many cases even vanishingly small forces can cause large changes in the dynamical system parameters, and such situations are intuitively associated with the concept of the instability. Obviously the destabilizing forces cannot be considered as "insignificant features, " and therefore, they cannot be ignored. But since they may be humanly indistinguishable in the? very beginning, there is no way to incorporate them into the model. This simply means that the model is not adequate for quantitative description of the corresponding dynamical process: it must be changed or modified.

However, the instability delivers an important qualitative? information: it manifests the boundaries of applic.ability of the original model.

We will distinguish short and long-term instabilities. Short-term instability occurs when the system has alternative stable states. For dissipative? systems such states can be represented by static or periodic attractors. In the very beginning of the post-instability transition period, the unstable motion cannot be traced quantitatively, but it becomes more and more deterministic as it approaches the attractor. Hence, a short-term instability does not necessarily require a model modification. Usually this type of imitability is associated with bounded deviation of position coordinates whose changes affect the energy of the system. Indeed, if the growth of a position coordinate persists, the energy of tile system would become unbounded.

The long term instability occurs when the system does not have an alternative stable state. Such type of instability can be associated only with ignorable coordinates since t hese coordinates do not effect the energy of the syst em. The long term instability is the main cause of chaos. It can occur in tht? form of orbital instabilit y, Hadamard's instability, Reynolds instability, etc. We will illustrate the concept of long-term instability by the orbital instability.

First we recall that a coordinate q_α is called ignorable if it does not enter the Lagrangian funct ion L while the corresponding non-conservative generalized forces Q_α or is zero:

$$\frac{\partial L}{\partial q_\alpha} = 0, \quad Q_\alpha = 0 \tag{60}$$

therefore,

$$\frac{\partial L}{\partial \dot{q}_\alpha} = P_\alpha = Const \tag{61}$$

it?., the generalized ignorable impulse P_α is constant.

As follows from Eq.(61), there exist such states of dynamical systems (called stationary motions) that all the position (i.e. non-ignorable) coordinates retain constant value while the ignorable coordinates vary in accordance wit h a linear law. For example, a regular precession of a heavy symmetric gyroscope is a stationary motion characterized by the equation:

$$\Theta = Const, \ \dot{\psi} = Const, \ \dot{\phi} = Const \tag{62}$$

where the angle of precession ψ and the angle of pure rotation ϕ are ignorable coordinates, while the angle of nutation Θ - an angle formed by the axis of gyroscope and the vertical is a position coordinate.

obviously, stationary motions are not stable with respect to ignorable velocities: a small change in $\dot{q}_\alpha$ at t = O yields, as time progresses, an arbitrarily large change in the ignorable coordinates themselves. However, since this change mcreases linearly (hut not exponentially), the motion is still considered as predictable. In particular, the Lyapunov exponents for stationary motions are zero:

$$\sigma = \lim_{d(0)\to 0\ ,\ t\to\infty} \left(\frac{1}{t}\right) 111\frac{d(0)t}{d(o)} = 0 \tag{63}$$

However, in case of nonstationary motions) the ignorable coordinate can exhibit more sophisticated behaviors. In order to demonstrate this, let us consider an inertial motion of a particle M of unit mass on a smooth pseudosphere S having a constant negative curvature:

$$G_o = Const < 0 \tag{64}$$

Remembering that trajectories of inertial motions must be geodesics of S, we will compare two different trajectories assuming that initially they are parallel and that the distance bet ween them, ϵ_0, is very small.

As shown in differential geometry, the distance between such geodesics will exponentially increase:

$$\epsilon = \epsilon_0 e^{\sqrt{-G_0}t},\ G_0 < 0 \tag{65}$$

Hence, no matter how small the? initial distance ϵ_0, the current distance ϵ tends to infinity.

Let us assume now that the accuracy to which the initial conditions are known is characterized by L. It means that any two trajectories cannot be distinguished if the distance between them is less than L, i.e. if:

$$\epsilon < L \tag{66}$$

The period during which the inequality (66), holds has the order:

$$\Delta t \sim \frac{1}{|\sqrt{-G_0}|}\ ln\frac{L}{\epsilon_0} \tag{67}$$

However, for

$$t >> At \tag{68}$$

these two trajectories diverge such that they can be distinguished and must be considered as two different trajectories. Moreover, the distance between them tends to infinity even if ϵ_0 is small (but not infinitesimal). That is why the motion, once recorded, cannot be reproduced again (unless the initial conditions are known exactly), and consequently, it attains stochastic features. The Lyapunov exponent for this motion is positive and const ant:

$$\sigma = \lim_{t\to\infty\ ,\ d(0)\to 0} \left(\frac{1}{t}\right) 111\frac{\epsilon_0 e^{\sqrt{-G_0}t}}{\epsilon_0} = \sqrt{-G_0} = \text{Const} > 0 \tag{69}$$

Let us introduce a system of coordinates at the surface S: the coordinate q_1 along the geodesic meridians, and the coordinate q_2 along the parallels. In differential geometry such a system is called semi-geodesical. The square of the distance between adjacent point on the pseudosphere is:

$$ds^2 = g_{11}\ dq_1^2 + 2g_{12}dq_1dq_2 + g_{22}dq_2^2 \tag{70}$$

where

$$g_{11} = 1, g_{12} = 0,\ \ g_{22} = -\frac{1}{G_0}e^{-2\sqrt{-G}q_1} \tag{71}$$

The Lagrangian for the inertial motion of the particle M on the pseudosphere is expressed via the coordinates and their temporal derivates as:

$$L = g_{ij}\dot{q}_i\dot{q}_j = \dot{q}_1^2 - \frac{1}{G_0}e^{-2\sqrt{-G}q_1}\dot{q}_2^2 \tag{72}$$

and, consequently,

$$\frac{\partial L}{\partial q_2} = 0 \tag{73}$$

while

$$\frac{\partial L}{\partial q_1} \neq 0, \text{if } \dot{q}_2 \neq 0 \tag{74}$$

Hence, q_1 and q_2 play roles of position and ignorable coordinates, respectively,

Therefore, an inertial motion of a particle on a pseudosphere is stable with respect to the position coordinate q_1, but it is unstable with respect to the ignorable coordinate. However, in contradistinction to the stationary motions considered above, here the instability is characterized by exponential growth of the ignorable coordinate, and that is why the motion becomes unpredictable. It can be shown that such a motion becomes stochastic[1].

Instability with respect to ignorable coordinates can be associated with orbital instability. Indeed, turning to the last example, one can represent the particle velocity v as the product :

In the course of the instability, the velocity magnitude $|\mathbf{v}|$, and consequently, the total energy, remain unchanged, while all the changes affect only τ, i.e. the direction of motion. In other words, orbital instability leads to redistribution of the total energy between the coordinates, and it is characterized by positive Lyapunov exponents.

The results described above were related to inertial motions of a particle on a smooth surface. However, they can be easily generalized to motions of any finite-degree-of-freedom mechanical system by using the concept of configuration space. Indeed, if the mechanical system has N generalized coordinates $q^1(i = 1,2,'..., N)$ and is characterized by the kinetic energy:

$$W = \alpha_{ij}\dot{q}^i\dot{q}^j \tag{75}$$

then the configurate ion space can be int roduced as an N- dimensional space with the following metric tensor:

$$g_{ij} = a_{ij} \tag{76}$$

while the motion of the system is represented by the motion of the unit-mass particle in this configuration space.

In order to continue the analogy to the motion of the particle on a surface in actual space we will consider only two-dimensional subspaces of the N-dimensional configuration spare?, without loss of generality. Indeed, a motion which is instable in any such subspace, has to be qualified as an unstable in the entire configuration space.

Now the Gaussian curvature of a t wo-dimensional configurat ion subspace (q^1, q^2) follows from the Gauss formula:

$$G = \frac{1}{a_{11}a_{22} - a_{12}^2}\left(\frac{\partial^2 a_{12}}{\partial q^1 \partial q^2} - \frac{1}{2}\frac{\partial^2 a_{11}}{\partial q^2 \partial q^2} - \frac{1}{2}\frac{\partial^2 a_{22}}{\partial q^1 \partial q^1}\right) - \Gamma_{12}^{\gamma}\Gamma_{12}^{\delta}\, a_{\gamma\delta} - \Gamma_{11}^{\alpha}\Gamma_{22}^{\beta} a_{\alpha\beta} \tag{77}$$

where the connection coefficients Γ_{sk}^{l} are expressed via the Christoffel symbols:

$$\Gamma_{sk}^{l} = \frac{1}{2}a^{lp}\left(\frac{\partial a_{sp}}{\partial q^k} + \frac{\partial a_{kp}}{\partial q^s} - \frac{\partial a_{sk}}{\partial q^p}\right) \tag{78}$$

while

$$a^{\alpha\beta}a_{\beta\gamma} = a_{\gamma}^{\alpha} = \begin{cases} 0 & \text{if } \alpha \neq \gamma \\ 1 & \text{if } \alpha = \gamma. \end{cases} \tag{79}$$

Thus, the Gaussian curvature of these subspaces depends only on the coefficients a_{ij}, i.e. it is fully determined by the kinematical structure of the system [see equation (75)]. In case of inertial motions, the trajectories of the representative particle must be geodesics of the configuration space. Indeed, as follows from (74):

$$\frac{d\tau}{dt} = \frac{d\tau}{ds}\dot{s} = 0 \text{ if } \dot{v} = 0, \text{ and } |v| = |\dot{s}| = Const \neq 0 \tag{80}$$

where s is the arc coordinate along the particle trajectory:

$$ds = a_{ij}dq^i dq^j \tag{81}$$

But then:

$$\frac{d\tau}{ds} = o \tag{82}$$

which is the condition that the trajectory is geodesic.

If the Gaussian curvature (77) which is uniquely defined by the parameters of the dynamical system a_{ij}, is negative:

$$G < 0 \tag{83}$$

then the trajectories of inertial motions of the system originated at close, but different points of the configuration space diverge? exponentially from each other, and the motion becomes unpredictable and stochastic. Some examples of orbital instability in inertial,potential and general motions as well as other types of instability are disc ussed by M. Zak [11].

Turning back to the motion of the particle M on a smooth pseudosphere (Fig. 2), let us depart from inertial motions and introduce a force F acting on this particle. For noninertial motions (F # O) the trajectories of the particle will not be geodesics, while the rat e of their deviation from geodesics is ch aract eri zed by the geodesic curvature χ. It is obvious that this curvature must depend on the forces F:

$$\chi = \chi(F) \tag{84}$$

L. Synge *(4)* has shown that f the force F is potential:

$$F = -\nabla \Pi \tag{85}$$

where n is the potential energy, then the condition (83) is replaced by the following:

$$G_0 + 3\chi^2 + \frac{1}{W}\left(\frac{\partial^2 \Pi}{\partial q^i \partial q^j} - \Gamma^k_{ij}\frac{\partial \Pi}{\partial q^k}\right)n^i n^j < 0; i, j = 1, 2 \tag{86}$$

Here Γ^k_{ij} are defined by Eqs. (78), and n^i are the contravariant components of the unit normal $\mathbf{n}$ to the trajectory.

The geodesical curvature χ in (86) can be expressed via tilt? potential force F:

$$\chi = 2\frac{\mathbf{F} \cdot \mathbf{n}}{W} = -\frac{\nabla \Pi \cdot \mathbf{n}}{2W} \tag{87}$$

As follows from (86) and (87), the condition (86) reduces to (83) if F = O.

Suppose for example, that the following elastic force:

$$F = -\alpha^2 \epsilon, \alpha^2 = Const \tag{88}$$

proportional to the normal deviation ϵ from the geodesic trajectory is applied to the particle M moving on the smooth pseudosphere. If the initial velocity is directed along one of the meridians (which are all geodesics), the unperturbed motion will be inertial, and its trajectory will coincide with this meridian sine.tl there $\epsilon = 0$, and therefore, F = O. In order to verify the orbital stability of this motion, let us turn to tile criterion (38). Since:

$$\chi = 0, \text{ and } \frac{\partial \Pi}{\partial q^k} = F^k = 0 \tag{89}$$

for the unperturbed motion, one obtains the condition for orbital stability:

$$G_0 + \frac{\alpha^2}{2W} > 0, \text{ i.e. } \alpha^2 < -2WG, \quad G < 0 \tag{90}$$

whine

$$\mathrm{w} = \frac{1}{2}mv_0^2 \tag{91}$$

As in the case of inertial motions, the inequality:

$$\alpha^2 < -2WG_0 \tag{92}$$

leads to unpredictable (stochastic) motions which are characterized by:

$$\sigma = \sqrt{G_0 - \frac{\alpha^2}{2W}} = \text{Const} > 0 \tag{93}$$

For pure inertial motions ($\alpha = 0$), Eq. (93) reduces to Eq. (64).

After the discovery of chaos, the stochastic motions which are generated by the instability and are characterized by positive Lyapunov exponents, are called chaot it.. Hence, the inequalities (83) and (86) can be associated with criteria of chaos: if the left hand part in (86) is bounded away from zero by a negative number -II in all the configuration space where the motion can occur, them the? motion will be chaotic, and its positive Lyapunov exponent will be:

$$\sigma \geq B^2 \tag{93}$$

Unfortunately, this criterion is too "strong" to be of practical significance: it is sufficient, but not necessary. Indeed, this criterion assumes that not only global, but also the local Lyapunov exponents are positive in any point of the configuration space, At the same time, for many chaotic motions, local Lyapunov exponents in cert ain domains of the configuration spare? are all negat ive, or zero, although some of t h e global exp onents are still posit ive.

Following J. L. Synge[4], the results for the orbital instability of inertial and potential motions for a system of material points can be generalized to arbitrary motions [11].

Thus, there are some domains of dynamical parameters where the motion cannot be predicted because of instability of the solutions to the corresponding governing equations. How can it be interpreted? Does it mean that the Newton's laws are not adequate'? Or is there something wrong with our mathemat i cal models? In order to answer these questions, we will discuss some general aspects of the concept of instability, and in particular, a degree to which it is an invariant of motion. We will demonstrate that instability is an attribute of a mathematical model rather than physical phenomenon, that it depends upon the frame of reference, upon the class of funct ions in which the motion is described, and upon the way in which the distances between the basic and perturbed solutions is defined.

Let us turn to orbital instability discussed above. The metric of configurat ion space where the finite-degree- of-freedom dynamical system with N generalized coordinates q^i ($i = 1, 2 \cdots. N$) is represented by a unit-mass particle, was defined by Eqs. (75) and (76). Now there are at least two possible ways to define the distance between the basic and disturbed trajectories. Following Synge [4], we will consider the distance in kinematical and in kinematico-statistical sense. In the first case the corresponding points on the trajectories are those for which time t has the same value. In the second case the correspondence between points on the basic trajectory C and a disturbed trajectory C* is established by the condition that P (a point on C) should be the foot of the geodesic perpendicular let fall from P^* (a point on C*) on C, i.e., here every point of the disturbed curve is adjacent to the undisturbed curve (regardless of the position of the moving particle at the? instant t). As shown by Synge, both definition of stability are invariant with respect to coordinate transformations, and in both cases the stability implies that the corresponding distance between the curves C and C'* remains permanently small.

It is obvious that stability in the kinematical sense implies stability in the kinematico-statical sense, but the: converse is not true. Indeed, consider the motion of a particle of unit mass on a plain? under the influence of a force system derivable from a potential:

$$\Pi = -x + \frac{1}{2} y^2. \tag{94}$$

Writing down the equations of mot ion and solving them, we get:

$$x = \frac{1}{2} t^2 + At + B \tag{95}$$

$$y = c \text{ Sin } (t + \alpha) \tag{96}$$

where A, B, C and D are constants of integration.

Let the undisturbed motion be:

$$x = \frac{1}{2} t^2 + t \tag{97}$$

$$y = 0 \tag{98}$$

The motion is clearly unstable in the kinematical sense. However, from tht? viewpoint of stability in the kinematico-statical sense, the distance bet ween corresponding points is:

$$P P^* = y = C \text{ Sin } (t + D) \tag{99}$$

remains permanently small if C is small. Hence, there is stability in the kinematico- statical sense.

Thus, the same motion can be stable in one sense, and unstable in mother, depending upon the way in which the distance between the trajectories is defined.

It should be noticed that in both cases, the metric of configuration space was the same (see Eqs. (75) and (76). However, as shown by Synge [4], for conservative systems, one can introduce a configuration space with another metric.

$$g_{mn} = (E - \Pi)\alpha_{mn} \tag{100}$$

whine α_{mn} are expressed by Eq. (75), and E is the total energy.

The system of motion trajectories here consists of all the geodesics of the manifold. The correspondence between points on the trajectories is fixed by the condition that the arc O* P^* should be equal to the? arc OP, where O and o* are arbitrarily selected origins on the basic trajectory and any disturbed one, respectively.

As shown by Synge, the problem of stability here (which is called stability in the action sense) is that of the convergence of geodesics in Riemannian spare. If two geodesics pass through adjacent points in nearly parallel directions, the distance between points on the geodesics equidistant from the respective initial points is either permanently small or not. If not, there is instability. It appears that stability in the action sense may not be equivalent to stability in the kinematico-statical sense for distances which change the total energy E.

Turning to the example, Eq. (94), let us take the initial point O at the origin of coordinates and the initial point O* on the y axis. Then the disturbance being infinitesimal, the (action) distance between corresponding points is:

$$P^* = (E - \Pi)^{1/2} y = 2^{-1/2}(t + 1) \; C \operatorname{Sin}(t + D) \tag{101}$$

Hence, the motion is unstable in the action sense.

Dynamical instability depends not only upon the metric in which the distances between trajectories are defined, but also upon the frame of reference in which the motion is described.

For instance, as noticed by Arnold [1], an inviscid stationary flow with a smooth velocity field (in Eulerian representation):

$$v_x = A \sin z + C \cos y, \; v_y = B \sin x + A \cos z, \; v_z = C \sin y + B \cos x \tag{102}$$

has chaotic trajectories x(t), y(t), z(t) of fluid particles (Lagrangian turbulence) due to negative curvature of the configuration space which is obtained as a finite-dimensional approximation of a continuum. Thus, the same motion is stable in the eulerian representation, but is unstable in the Lagrangian one.

In order to demonstrate the instability dependence upon the class of functions in which the motion is considered, start with an example of a vertical ideally flexible inextensible string with a free lower end suspended in a gravity field. The governing equation for small transverse motion of the string is:

$$\frac{\partial^2 x}{\partial t^2} + \frac{T}{p}\frac{\partial^2 x}{\partial \psi^2} = 0 \tag{103}$$

It has the following characteristic speeds of the transverse waves propagation:

$$\mu = \pm\sqrt{\frac{T}{\rho}} \tag{104}$$

Since the tension of the string T vanish at the free end:

$$T = 0 \text{ at } S = l \tag{105}$$

where l is the length of the string, the characteristic speeds (104) vanish too at $S = l$, and therefore, Eq. (103) degenerate from hyperbolic into parabolic type at the very end of the string.

Suppose that an isolated transverse wave of small amplitude was generated at the point of suspension. The speed of propagation of the leading front of the transverse wave will be smaller than the speed of the trailing front because the tension decreases from the point of suspension to the free end. Hence, the length of the? above wave will be decreasing and in some cases will tend to zero. then according to the law of conservation of energy, the specific kinetic energy per unit of length will tend to infinity producing a map (snap of a whip).

As shown by M. Zak [11], a formal mathematical solution to Eq. (103) is stable in the? open interval (which does not include the end):

$$0 \leq x < \ell$$

but it is unstable in the closed interval:

$$0 \leq x \leq \ell$$

However, the. stable solution does not describe the snap of the whip, while the unstable solution does!

Thus, the properties of solutions to differential equations such as existence, uniqueness and stability, have a mathematical meaning only if they are referred to a certain class of functions, Most of the results concerning the properties of solutions to differential equations require differentiability (up to a certain order) of the functions describing the solutions. However, the mathematical restrictions imposed upon the class of functions which guarantee the existence of an unique and stable solution, do not necessarily lead to the best representation of the corresponding physical phenomenon. Indeed, turning again to Eq. (103), one notices that the unique and stable solution does not describe a cummulation effect (a snap of a whip) which is well pronoum cccl in experiments. At the same time, an unstable solution in a closed interval gives a qualitative description of this effect. Hence, pure mathematical restrictions imposed upon the solutions are not always consistent, the long-term instability in classical dynamics discussed above, can be interpreted as a discrepancy between these mathematical restrictions and physical reality. This means that unpredictability in classical dynamics is a price paid for mathematical " convenience" in dealing with dynamical models. Therefore, the concept of unpredictability in dynamics should be put as unpredictability in a selected class of functions, or in a selected metrics of configuration space, or in a selected frame of reference.

In this connection one should notice that the governing equations of classical dynamics, and in particular, of continuous systems, in addition to Newton's laws, are based upon a pure mathematical assumption that all the functions describing the system motions, must be differentiable "as many times as necessary". But since this assumption is not always consistent with the physical nature of motions, such an inconsistency leads to instability (in the class of smooth functions) of the governing equations[11].

Hence, the occurrence of chaos or turbulence in description of mechanical motions means only that these motions cannot be properly described by smooth functions if the scale of observations is limited. These arguments can be linked to Godel's incompleteness theorem[3], and the? Richardson's[7] proof that the theory of elementary functions in classical analysis is undecidable.

Thus, since instability is not an invariant of motions, the following question can be posed: is it possible to find such a new (enlarged) class of functions, or a new metric of configuration space, or a new frame of reference in order to eliminate instability? Actually such a possibilit y would lead to different represen tat ive paramet ers describing the same mot ion in'such a way that small uncert aint ies in ext ernal forces cause small changes of these parameters. For example, in turbulent and chaot ic mot ions, mean velocit it?s, Reynolds stresses, and power spectra, represent "st able" parameters, although classical governing equations neither are explici tly expressed via these paramet ers, nor uniquely define them.

The first step toward the enlarging of the class of functions for modeling turbulence was made by O. Reynolds (1895) [6] who decomposed the velocity field into the mean and pulsating components, and actually introduced a multivalued velocity field. However, this decomposi tion brought new unknowns without addit ional governing equations, and that created a " closure" "problem. In 1986 Zak[1 ı] has shown that the Reynolds equations can be obtained by referring the Navier-Stokes equations to a rapidly oscillating frame of reference, while the Reynolds stresses represent t he cont ribut ic m of inertia forces. From this viewpoint the "closure" has the same status as "proof" of Euclid's parallel postulate, since the motion of the. frame of reference can be chosen arbitrarily. In other words, the "c1osuıw" of Reynolds equations represents a case of undecidabilit y in classical mechanics. However, based upon the interpretation of the Reynolds stresses as inertia forces, it is reasonable to choose the motion of the frame of reference such that the inertia forces eliminate. the original imitability. In other words, the enlarged class of functions should be selected such that the solution to the original problem in that class of functions will not possess an exponential sensitivity to changes in initial conditions. This stabilization principle has been formulated and applied to chaotic and turbulent motions by Zak [1 1]. As shown there, the motions which are chaot ic (or turbulent) in the original frame of reference can be represented as a sum of the "mean" motion and rapid fluctuations, while both components are uniquely defined. It is worth emphasizing that the amplitude of velocity fluctuation is proportional to the degree of the original inst abilit y, and therefore, the rapid fluctuations can be associated with the measure oft he uncertaint y in the description of the motion. It should be not iced that bot h " mean" and " fluct uat ion" components representing the originally chaotic motion are stable, i.e., they are not sensitive to changes of initial conditions, and are fully reproducible.

7. CHAOS IN FAST OSCILLATING FRAME OF REFERENCE

Formally, chaos is caused by instability of trajectories (orbital instability). If the velocity of a particle is decomposed as $\vec{v} = v\vec{\tau}$, ($\vec{\tau}$ is the unit vector along the trajectory), then orbital instabilities are identified with instabilities of $\vec{\tau}$. In other words, the orbital instability leads only to redistributions of the energy between different coordinates, and it can be associated with an ignorable variable which does not contribute into kinetic energy. Therefore, an unlimited growth of this variable does not violate the boundedness of energy. That is wily the orbital instability may not lead to classical attractors and chaos can emerge . In dissipative systems the persisting instability can be "balanced" by dissipative forces in a sense that exponentially diversing trajectories are locked up within a cont ratting phase-space volume, and this leads t o chaot ic attract ors. In both conservative and dissipative systems, exponential divergence of trajectories within a constant or a contracting volume causes their mixing, so that the motion cannot be traced unless the initial conditions are known to infinite accuracy. It means that in configurat ion space, two different trajectories which may be initially indistinguishable (bee.aus; of finite scale of observat ion), diverge exponent idly, so t hat a "real" t raject ory can fill up all t he spacing bet ween these exponentially diverging trajectories. In other words, in the. domain of exponential inst ability, each trajectory "multiplies", and therefore, the predicted trajectory become mult ivalued, so the velocit ies can be considered as ran dom variables:

$$\dot{q}^i = \dot{q}^i(t, \varepsilon), \ 0 \leq \varepsilon \leq 1 \tag{106}$$

where $\dot{q}$ and ε for a fixed t are a function and a point on a probability space, respectively. Let us refer the original equations of motions to a mm-inertial frame of reference which rapidly oscillate? with respect to the original inertial frame of reference. Then the absolute velocity q can be decomposed into the relative velocity $\dot{q}_1$ and the transport velocity $\dot{q}_2 = 2\dot{q}_{2(o)}$:

$$\dot{q} = \dot{q}_1 + 2\dot{q}_{2(o)} \cos \omega t, \quad \omega \to \infty \tag{107}$$

while $\dot{q}_1$ and $\dot{q}_2^o$ are "slow" functions of time in the sense that

$$w >> \frac{1}{\tau} \tag{108}$$

where τ is the time scale upon which the changes q_1 and $\dot{q}_{2(o)}$ can be ignored.

Then for the mean $\bar{q}$:

$$q \cong q_1 \text{ since } \int_o^{t \gg \tau} \dot{q}_{2(o)} \cos \omega t dt \simeq \frac{1}{\omega} \dot{q}_{2(o)} \sin \omega t \to 0 \text{ if } \omega \to \infty \tag{109}$$

In other words, a fast oscillating velocity practically does not change the displacement s .

Taking into account that

$$\frac{\omega}{2\pi} \int_o^{2\pi/\omega} \dot{q}_1 dt \simeq \dot{q}_1, \int_0^{2\pi/\omega} \dot{q}_{2(o)} \sin \omega t dt = O, \int_0^{2\pi/\omega} \ddot{q}_{2(o)} \cos \omega t dt = o \tag{110}$$

$$\text{and} \quad \int_o^{2\pi/\omega} \dot{q}^2_{2(o)} \cos^2 \omega t dt = \frac{1}{2} \dot{q}^2_{2(o)}$$

one can transform a system:

$$\dot{x}^i = a^i_j x^j + b^i_{jm} x^i x^m, i = 1, 2, \cdots n \tag{111}$$

into the following form:

$$\bar{x}_i = a^i_j \bar{x}^j + b^i_{jm} \bar{x}^j \bar{x}^m, + b^i_{jm} \overline{x^j x^m}, i = 1, 2, \cdots n \tag{112}$$

where $\bar{x}^i$ and $\overline{x^i x^j}$ are means and double- correlat ions of x^i as random variables, respectively.

Actually the transition from (111) to (1 12) is identical to the Reynolds transformation: indeed, being applied to the Navier-St ekes equations, it leads to the Reynolds equations, and therefore, the last terms in (112) (which is a contribution of inertial forces due to fast oscillations of the frame of reference) can be identified with the Reynolds stresses. From a mathematical viewpoint, this transformat ion is int erpretable as an enlarging the class of smooth funct ions to mult ivalued ones. haled, as follows from (108), for any arbitrarily small interval At, there always exists such a large frequency w > $\Delta t/2\pi$ that within this interval the veloci t y $\dot{q}$ runs t hrough all its values, and actually the velocity field becomes multivalues.

The most significant advantage of the Reynolds-type equations (112) is that they are explicitly expressed via the physically reproducible parameters $\bar{x}$-", $x^i x^j$ which describe, for instance, a mean velocity profile in turbulent motions, or a power spectrum of chaotic attractors. However, as a price for that, these equations require a closure since the number of unknowns in there is larger than the number of equations. .4ctually the closure problem has existed for almost hundred years since the Reynolds equations were derived. In the next sections, based upon the stabilization principle introduced by Zak, M. [1 1] this problem will be discussed.

Some comments should be made concerning the Reynolds transformation of the Lagrange equation (59). Their explicit form:

$$\ddot{q}^r + \Gamma^r_{mn}\dot{q}^m\dot{q}^n = Q^r, \quad Q^r = -\frac{\partial \Pi}{\partial q^r} \tag{113}$$

in general, is nonlinear with respect to both the coordinates q^r and the velocities $\dot{q}^r$ since

$$\Gamma^r_{mn} = \Gamma^r_{mn}(q_1, \cdots q_n), \Pi = \Pi(q_1, \cdots 1_n) \tag{114}$$

However, as follows from Eqs. (107), the fluctuations of the coordinates are much smaller than tht? fluctuations of the velocities:

$$q_{2(0)} \sim \frac{1}{\omega}\dot{q}_{2(0)}, \quad \omega \to \infty, \tag{115}$$

and therefore, they can be ignored.

Consequently, after the Reynolds transformation, Eq. (113) are presented in the form:

$$\bar{q}^r + \Gamma^r_{mn}\bar{q}^m\bar{q}^n = Q^r + Q^r_{(i)}, \quad Q^r_{(i)} = \Gamma^r_{mn}\overline{\dot{q}^m\dot{q}^n} \tag{116}$$

where $\bar{q}^r = q^r_1$ is the mean value of the coordinate q^r, and $\overline{\dot{q}^m\dot{q}^n}$ is the averaged product of the flue.tuatim velocities, and the Reynolds force $Q^r_{(i)}$ represents the contribution of inertia caused by the transport motion of the frame of reference.

Actually the transformation from (113) to (116) can be based upon the axiomatically introduced Reynolds conditions:

$$\overline{a+b} = \bar{a} + \bar{b}, \quad \overline{ab} = \bar{a}\bar{b} + \overline{a'b'} \quad \text{if} \quad a = \bar{a} + a', \text{etc.} \tag{117}$$

In a particular case

$$\Gamma^r_{mn} \equiv 0 \tag{118}$$

i.e., when the configuration space is Euclidean,

$$Q^r_{(i)} \equiv 0 \tag{119}$$

and the nonlinearities of coordinates cannot be ignored any more. Representing q^r in the form:

$$Q^r(\bar{q} + q') = Q^r(\bar{q}) + Q^r_{(i)}, Q^r_{(i)} = [Q^r(\bar{q} + q') - Q^r(\bar{q})] \tag{120}$$

one obtains instead of Eq. (1 16):

$$\bar{q}^r = Q^r(\bar{q}_1, \cdots \bar{q}_n) + \overline{Q}^r_{(i)} \tag{121}$$

8. STABILIZATION PRINCIPLE

The main purpose of the transition from the form (101) to the form (102) is to change the representative parameters describing the motion in such a way that they become physically reproducible, i.e., mathematically stable. Hence, the next logical step is to utilize the extra-variables, i.e., the Reynolds stresses, for elimination of the original instability. bother words, one can seek such an additional relationships:

$$\varphi(\overline{x^i x^j}, \bar{x}^i, \bar{x}^j, \cdots) = 0 \tag{122}$$

which makes the system (1 12), (122) stable. Obviously, in this posedness of the problem, the solution to the system (112), (122) is not unique: the? system can be overstablized to any degree, while each of these stable solutions will have physical meaning. But for the best solution one has to minimize the uncertainties represented by the Reynolds stresses, and therefore, the system should be brought to the boundary of instability. Since the orbital instability causing chaos is characterized by positive Lyapunov exponents λ_i^+, One should select the Reynolds stresses in (1 12) such that

$$\lambda_i^+ = 0 \tag{123}$$

while keeping the rest of the Lyapunov exponents without changes:

$$\lambda_i^o = \tilde{\lambda}_i^o, \lambda_i^- = \tilde{\lambda}_i^- \tag{124}$$

where $\lambda_i^o, \lambda_i^-, \tilde{\lambda}_i^-$ and $\tilde{\lambda}_i^-$ are non-positive Lyapunov exponents of the system (112), (122) and equation (111), respectively.

Clearly, those components of the Reynolds stresses which do not affect the Lyapunov exponents, must be omitted. In general, the solution to equations (102)-(105) will eventually approach a set of periodic attractors which "replaces" the chaotic attractor of equation (101). However one should consider these sets not as an approximation to the original chaotic attractor, but rather as a different way of mathematical representation of the same physical phenomenon. This representation is provided by a new frame of reference whose oscillations are coupled with the dynamical variables such that the inertia forces (i.e. the Reynolds, stresses) generated by transport motion, eliminate the original instability. In other words, the new frame of reference provides the best "view" of the motion.

The decomposition (102) applied to equation (101), generates not only pair correlations $\overline{x^i x^j}$, but also correlations of higher order, such as triple correlations $\overline{x^i x^j x^k}$, quadruple correlations $\overline{x^i x^j x^k x^m}$, etc. Indeed, multiplying equation (101) by x^k and averaging and combining the results, one obtains the governing equations for the pair correlations $\overline{x^i x^k}$;

$$\begin{aligned} \overline{x^i x^k} =& a^i_j \overline{x^j x^k} + a^k \overline{x^j x^i} + b^i_{jm}(\overline{x^k x^j x^m} + \overline{x^k x^j}\bar{x}^m + \overline{x^k x^m}\bar{x}^j) \\ &+ b_{jm}(\overline{x^i x^j x^m} + \overline{x^i x^j}\bar{x}^m + \overline{x^i x^m}\bar{x}^j), \end{aligned} \tag{125}$$

which contain nine additional triple correlations $\overline{x^i x^j x^k}$.

Now the application of the stabilization principle will lead to the system (112)-(125) which will define $\bar{x}^i, \overline{x^i x^j}$ and those components of triple corrections $\overline{x^i x^j x^m}$ which affect the Lyapunov exponents in equations (123) and (124). Hence, the solutions to the systems (1 12)-(124) and (1 12)-(125) can be regarded as the first and the second approximation, respectively, to the problem. Theoretically speaking, by considering next order approximations, a complete probabilistic structure of the solution to equation (111) can be reproduced.

Applications of the stabilization principle is significantly simplified for those systems whose boundaries of instability can be formulated analytically. Fox some cases of conservative chaos and simple turbulent flows new representations of solutions were given by Zak, M.[11].

In the next section we will demonstrate application of the stabilization principle to some dissipative chaotic systems by numerical elimination of positive local Lyapunov exponent s.

9. APPLICATION OF THE STABILIZATION PRINCIPLE TO REPRESENTATION OF CHAOS

a. Inertial Motions

In order to clarify the main idea of the approach, let us turn to the inertial motion of a particle M of unit mass in a smooth pseudosphere S having a constant negative curvation (64). As shown there, the orbital instability, and therefore, the chaotic behavior of the particle M can be eliminated by the? elastic force (88).

$$F = -\alpha^2\epsilon \ , \alpha^2 = \text{const.} > -2WG, \ G < 0 \tag{126}$$

proport ional to the normal deviat ion ϵ from the geodesic t raject ory which is applied to the particle M. But such a force can appear as an inertial force if the motion of the particle M is referred to an appropriate non-inertial system of coordinates.

Indeed, so far this motion was referred to an inertial system of coordinates q_1, $q2$, where q_1 is the coordinate along the geodesic meridians, and q_2 is the coordinate along the parallels. Let us introduce new a frame of reference which rotates clout the axis of symmetry of the pseudosphere with the readily oscillatory transport velocity:

$$\dot{\epsilon} = 2\dot{\epsilon}_0 \cos \omega t, \quad \omega \to \infty \tag{127}$$

so that the components of the resultant velocity along the meridians and parallels are, respectively:

$$v_1 = \dot{q}_1, \ v_2 = \dot{q}_2 + 2\dot{\epsilon}_0 \cos \omega t \tag{128}$$

since Eq. (128) has the same structure as Eq. (107), the Lagrangion of the motion of the particle M relative to the new (non-inertial) frame of reference can be written in the following form: (see Eq. 72):

$$L^* = \dot{q}_1^2 - \frac{1}{G_0} e^{-2\sqrt{-G_0}q_1}(\dot{q}_2^2 + \dot{\epsilon}_0^2) \tag{129}$$

The last term in Eq. (252) represents the contribution of the inertia forces in the new frame of reference.

So far the transport velocity $\dot{\epsilon}_0$ was not specified, and therefore, the Lagrangion (129) has the same element of arbitrariness as the governing equations (112) describing chaotic motions.

Now, based upon the stabilization principle, we are going to specify the transport motion in such a way that the original orbital instability of the inertial motion of the particle M is eliminated. Turning to the condition (90), one obtains:

$$\frac{\partial^2 L}{\partial \epsilon^2} \geq -2WG_0 \tag{130}$$

where $W = \frac{1}{2}v_0^2$ is the kinetic energy of the particle. This condition can be satisfied if the transport velocity $\dot{\epsilon}_0$ is coupled with the: normal deviation ϵ as following:

$$-\frac{1}{G_0}e^{-2\sqrt{-G_0}q_1}\dot{\epsilon}_0^2 = -WG_0q_2^2 2 \tag{131}$$

As follows from Eq. (93), in this limit case the Lyapunov exponent of the relative motion in the new (non-inertial) frame of reference will lx I zero:

$$\sigma = \sqrt{-G_0 - \frac{\bar{\partial}^2}{W}} = 0, \quad \partial^2 = \frac{\partial^2 L}{\partial \epsilon^2} \tag{132}$$

and the trajectories of perturbed motions do not diverge. The normal deviation from the trajectory of tht? relative motion (in case of zero perturbed velocity $\dot{\epsilon}_0$) can be written in the following form:

$$q_2 = q_2^0 = \text{Const}, \quad q_2^0 = q_2(t = o) \tag{133}$$

which means that in t he new frame of reference an i ni t ial error ϵ_0 does not grow - it remains const ant. The relat ive mot ion along the t raject ory is described by the different ial equation following from the? Lagrangion (120) which takes the following form (after substituting Eq. (131).

$$L = \dot{q}_1^2 - \frac{1}{G_0}e^{-2\sqrt{-G_0}q_1}\dot{q}_2 - WG_0q_2^0 \tag{134}$$

i.e.,

$$\ddot{q}_1 - \frac{2\sqrt{-G_0}}{G_0}e^{-2\sqrt{-G_0}q_1}\dot{q}_2 = 0 \tag{135}$$

But the original (unperturbed) motion was directed along the meridians, i.e., $\dot{q} \equiv 0$. Consequently,

$$\cdots q_1 = 0, \quad \dot{q}_1 = \bar{v}_0 = \text{Const} \tag{136}$$

i.e., the relative motion along the trajectory is constant.

However, this velocit y is different from the original velocit y V.. Indeed, the total kinetic energy of the particle now consists of the kinetic energy of the motion along the trajectory, and the kinetic energy of transverse fluctuations expressed by Eq. (131), i.e.

$$\frac{v_0^2}{2} = \frac{\bar{v}_0^2}{2} + \frac{v_0^2}{2}(q_2^0)^2 \mid G_0 \mid \tag{137}$$

whence:

$$\bar{v}_0 = v_0\sqrt{1 - (q_2^0)^2 \mid G_0 \mid} < v_0 \tag{138}$$

Thus, the original unstable (chaotic.) motion is decomposed into the mean motion along the trajectory q_2 = const with the? constant velocity (129), and transverse fluctuations whose kinetic energy is proportional to the original error q_2^0 and to the degree Of instability $|G_0|$. It is important to emphasize that both components of the motion are stable in a sense that initial error in q_2 at t = O does not grow, and initial error in q_1 at t = O *grows* linearly with time.

Obviously the mean, or averaged motion represents a macroscopic view on the particle behavior extracted from the microscopic world, while the irreversibility of this motion is manifested by the loss of the initial kinetic energy to microscopic flue.tuaticms.

It should be emphasized that the decomposition of the motion into regular and fluctuation components was enforced by the stabilization principle as a supplement to Newtonian mechanics (see Eq. (131)), while without this principle any theory where dynamical instability can occur is incomplete.

b. Potential Motions

Based upon Eqs. (116), for potential motions, the governing equations can be written in the following form:

$$\ddot{q}^\alpha + \Gamma^\alpha_{\beta\delta}\dot{q}^\beta\dot{q}^\delta = -\frac{\partial\Pi}{\partial q^\alpha} + a^\alpha_{(i)} \tag{139}$$

$$\frac{\partial\Pi}{\partial q^\alpha} = -Q^\alpha \tag{140}$$

where n is the pot ent ial energy of the dynamical system, and $Q^\alpha_{(i)}$ are the inertia forces (or the " Reynolds stresses" caused by the rapidly oscil lating transport motion of the frame of reference.

For simplicit y, we will confine ourselves by a t wo-dimensional dynamical system assuming that α = 1, 2.

Following the same strategy as those applied to inertial motions, let us couple the inertia forces with tht? parameters of the dynamical system in such a way that the original orbital instability (if it occurs) is eliminated. For that purpose, first we will represent this forces in the form:

$$Q^\alpha_{(i)} = -\frac{\partial\Pi_{(i)}}{\partial q^\alpha} \tag{141}$$

where $\Pi_{(i)}$ is a fictitious potential energy equivalent to the kinetic energy of the fluctuations. Then, turning to the criteria of local orbital stability (86), one finds this potential energy $\Pi_{(i)}$, and consequently, the inertia forces $Q^\alpha_{(i)}$ from the condition that original local orbital instability is eliminated:

$$G + 3\left[\frac{\nabla(\Pi + \Pi_{(i)})\bullet n}{2W}\right]^2 + \frac{1}{2W}\left[\frac{\partial^2(\Pi + \Pi_{(i)})}{\partial q^i \partial q^j} \cdot \Gamma^k_{ij}\frac{\partial(\Pi + \Pi_{(i)})}{\partial q^k}\right] n^i n^j = 0 \quad i, j = 1,2. \tag{142}$$

Here W, G, and Γ^k_{ij} are defined by the parameters of the dynamical system (1 16) via Eqs. (75), (77) and (78), respectively, and n_i are the contravariant components of the unit normal n to the trajectory of the basic func tion.

Eq. *(142)* contains only one unknown $\Pi_{(i)}$ which can be found from it, and that will define the inertia forces, or the "Reynolds stresses" (141)

It should be noticed that unlike the case of the inertial motion of a particle on a pseudosphere, here? the G oeussian curvat ure G, as well as the gradients of the potential energy Π, are not constants, and consequently, the local Lyapunov exponents may be different from the global ones. This means that the condition (142) eliminates local positive exponents, and therefore, the solution to Eqs. (139) and (142) represents an over stabilized mot ion. Obviousl y, eliminat ion of only global posit ive L y apunov exponents would lead to solutions with less uncertainties while some of local exponents in certain domains of the phase space may even remain positive. However, the strategy for elimination of global positive exponents is more sophisticated, and it can be implemented only numerically.

It is worth noting that Eq. (142) is simplified to the following:

$$G + \frac{1}{2W}\left[\frac{\partial^2(\Pi + \Pi_{(i)})}{\partial q^i \partial q^j}\right] n^i n^j = 0, \tag{143}$$

if the basic motion is characterized by zero potential forces

$$\frac{\partial \Pi}{\partial q^i} = 0 \tag{144}$$

It may occur, for instance, when the dynamical system is in a relative equilibrium with respect to a moving frame.

Thus, as in the previous case of inertial motion of a particle, here the Lagrange equations (139) are supplemented by t he addit ional const raint (142) following from the stabilization principle. It is important to emphasize? that this constraint is effective only in case of orbital instability of Eq. (139); otherwise it is satisfied automatically.

As an illustration to the case of potential system, we will consider the motion of a charged particle (charge –e, mass m) in a uniform magnetic field, B in the vicinity of a metallic sphere (radius a) biased to a potential $\mathbf{V}_0 > 0$:

$$m\dot{\mathbf{v}} = -e\,\mathbf{v} \times \mathbf{B} + e\,\nabla \mathbf{v} \tag{145}$$

where $v = \frac{d\hat{r}}{dt}$ is the velocit y of the particle, and $v = v_0(a/p)$ is the electrical potential due to the sphere.

Eq. (145) can be written in a dimensionless form:

$$\dot{v}_x = -\frac{x}{r^3} v_y, \quad \dot{v}_y = -\frac{y}{r^3} + v_x, \quad \dot{v}_z = -\frac{z}{r^3}, \tag{146}$$

$$v_x = \dot{x}, \quad v_y = \dot{y}, \quad v_z = \dot{z}, \tag{147}$$

where

$$\dot{x} = \frac{dx}{d\tau}, \cdots etg, \; r^2 = x^2 + y^2 + z^2$$

$$r = \frac{\hat{r}}{\lambda}, \quad \tau = \omega_e t, \quad \lambda^3 = ev_0 a/m\omega_e^2, \quad w = eB/m$$

As reported in [2], there are certain domains of init ial conditions which lead to chaotic trajectories. The system is chaotic, for instance at $x = 1.5$; $y = 0$, $z = 4.0$, $x_x = v_y = v_z = 0$ at $t = 0$. We have? reproduced these resul ts (see Fig. 1) by solving Eqs. (146), (147) numericall y.

The implementation of the stabilization principle, i.e. simultaneous solution of Eqs. (146) and (147) (after their Reynolds decomposition into the form (139)) and the constraint (142) were performed numerically. The numerical strategy was very simple: along with the basic solution, a perturbed solution were calculated and compared with the basic one after certain tire? steps; if the perturbed solution diverged faster than prescribed time - polynomial, then an appropriate Reynolds force was applied to suppress it; otherwise no actions were taken. The resulting trajectories in tile same x, y, z - phase space are plotted in Fig. 2. These trajectories represent an averaged, or expected motion which is not chaotic any more. It is important to emphasize that this motion is stable in the sense that small changes of the initial conditions will cause small changes in the motion.

Actually this example elucidates the mechanism of transition from the Hamiltonian mechanics describing fully reversible mechanical processes on the microscopic level, to irreversible macroscopic mot ions describing t berm dynamical l)rocesses. On the. same line of argumentation, the stabilization principle implements the preference to more probable states of the system over the less probable states.

DISCUSSION AND CONCLUSION

The problem of irreversibility in thermodynamics was revisited and analyzed on the microscopic, stochastic, and macroscopic levels of description. It was demonstrated that Newtonian dynamics (as well as any dynamical theory where chaotic solutions are possible) can be represented in the Reynolds form when eat.] ı dynamical variable is decomposed into the mean and fluctuation components. Additional equations coupling fluctuations and the mean values follow from the stabilization principle formulated in [11] and briefly described in the previous sections. Tilt? main idea of this print.iplt? is that the fluctuations must be selected from the condition that they suppress the original instability down to a neutral stability. Supplemented by the stabilization principle, the Hamiltonian, or Largranian formalisms can describe the transition from fully reversible? to irreversible motions as a result of the decomposition of chaotic motions (which are very likely to occur in many - body problems) into regular (macroscopic) motions and fluctuations. Actually the stabilization principle implements the preference to more probable states of the system over the less probable states, and from that viewpoint it can be associated with the averaging procedure exploited in statistical mechanics. However, the averaging procedure was always considered as an "alien intrusion" into the classical mechanics, and that caused many discussions around the? problem of irreversibility on the macroscopic level. On the contrary, the stabilization principle is a part of Newtonian mechanics (as well as a part of any dynamical theory where chaotic motions can occur), and therefore, it provides formal mathematical explanation for the transition from fully reversible to irreversible processes.

On the stochastic level of description, new phenomenological force with non-Lipschitz properties is introduced. This force as a resultant of a large number of collisions of a selected particle with other particles, has characteristics which are uniquely defined by the thermodynamical parameters of the process under consideration, and it represents a part of the mathematical formalism describing random-walk-like processes without invoking any probabilistic arguments.

Additional non-Lipscitz thermodynamical forces were incorporated into macroscopic models of transport phenomena in order to introduce a time scale. These forces art? effective only within a small domain around equilibria. Without causing any changes in other domains, they are responsible for finite time of approaching equilibria. Such a property is very important, for interpretation of irreversibility on the macroscopic scale. Indeed, there is always an extremely small (lint non-zero) probability that a particle performing a random walk can return to its original position passing through all of its previous steps backward, and therefore, this effect should not be excluded from the solutions to the macroscopic equations if they are observed during infinitely large period of time. However, these practically unrealistic situations may be excluded from the consideration in case of the modified macroscopic equations since they are characterized by a limited time sale,

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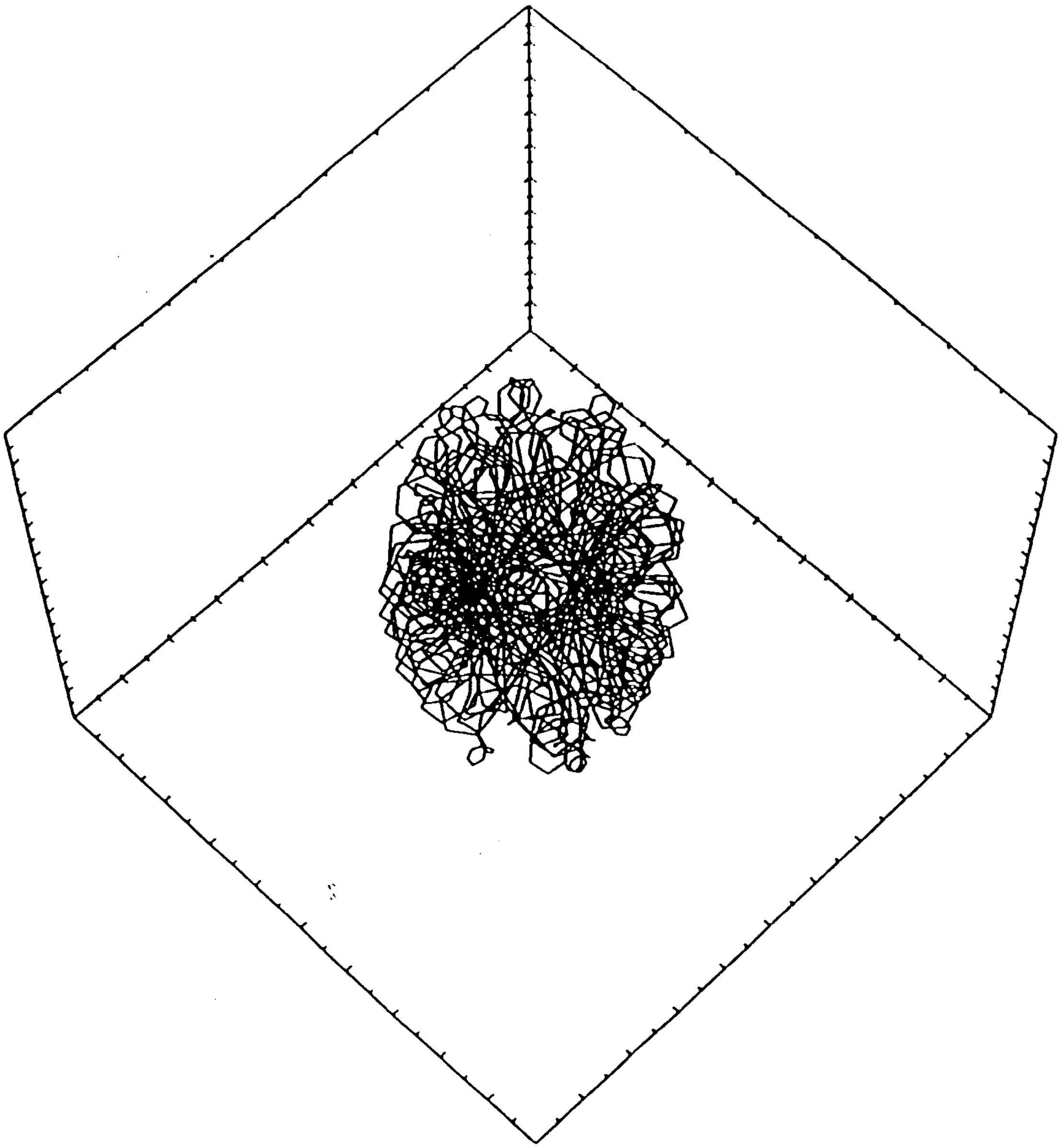

Figure 1.

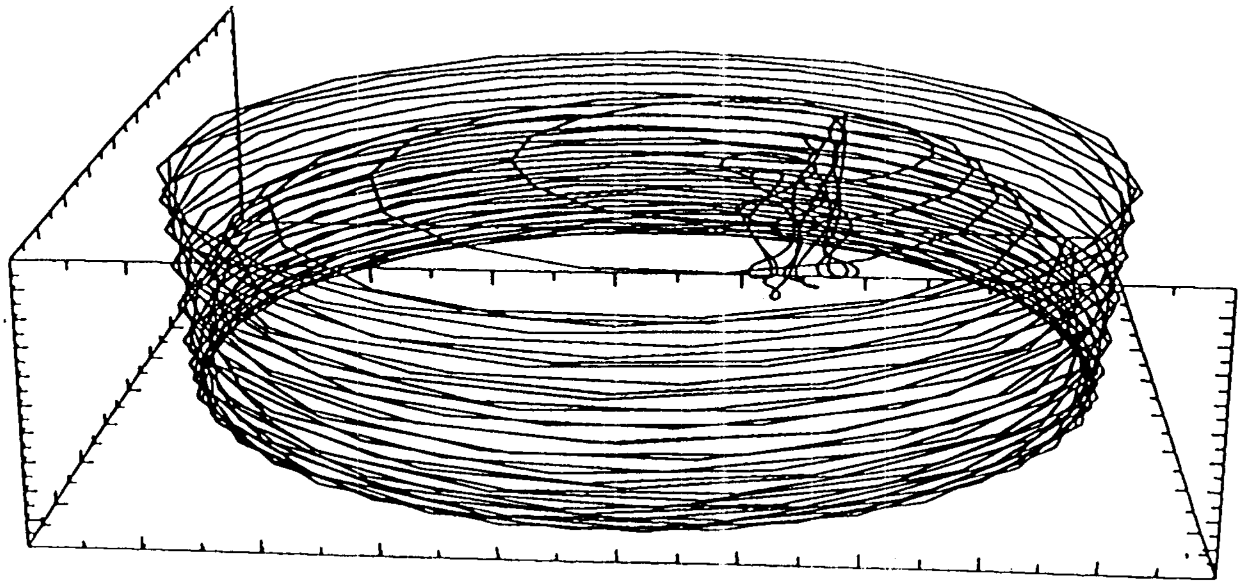

Figure 2